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# Robust controllability for linear uncertain descriptor systems

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**Abstract**

In this paper, under assumptions that the linear nominal descriptor system is regular and controllable, some sufficient conditions are proposed to preserve the assumed properties when both structured (elemental) and unstructured (norm-bounded) parameter uncertainties are added into the nominal descriptor system. Besides, another sufficient conditions are also presented to preserve the assumed properties for a class of linear descriptor systems having structured uncertainties in the structure information matrix as well as having both structured and unstructured parameter uncertainties in the system matrix and the input matrix simultaneously. The corresponding results for the dual observability robustness problems are straightforward extensions. Three numerical examples are given to illustrate the applications of the proposed sufficient conditions, and it is shown that the proposed sufficient conditions could be less conservative than the existing ones reported recently in the literature.



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## 1. Introduction

In recent years, the controllability problems of linear descriptor systems have attracted some attention in the literature (for example, see [1,5,7–9,11–14] and references therein) due to the significance of descriptor systems in both theory and applications. Sometimes the descriptor system is called singular system, generalized state-space system, implicit system, or semistate system [6]. To the authors' best knowledge, only Lin et al. [7–9] and Chou et al. [1] studied the robust controllability problems of linear descriptor systems. That is, the research on the robust controllability of linear descriptor systems is considerably rare and almost embryonic.

Lin et al. [7] proposed some sufficient conditions for robust controllability of linear descriptor systems with unstructured (norm-bounded) parameter uncertainties, where the unstructured parameter uncertainties are in the system matrix only. Lin et al. [8] studied the robust C-controllability (complete controllability) problem for linear uncertain descriptor systems with all matrices (structure information matrix, system matrix, and input matrix) being interval matrices. Lin et al. [9] also studied the robust controllability problems of linear descriptor systems with structured (elemental) parameter uncertainties in the system matrix and the input matrix. Based on the structured singular value approach, the sufficient conditions proposed by Lin et al. [8,9] are obtained by transforming the robust controllability problems into checking the nonsingularity of a class of uncertain matrices. On the other hand, it is well known that an approximate system model is always used in practice and sometimes the approximation error should be covered by introducing simultaneously both structured (elemental) and unstructured (norm-bounded) parameter uncertainties in control system analysis and design. That is, it is not unusual that at times we have to deal with a system consisting of two parts: one part has only the structured parameter uncertainties, and the other part has the unstructured parameter uncertainties. Therefore, Chou et al. [1] investigated the robust controllability problems of linear descriptor systems with both structured (elemental) and unstructured (norm-bounded) parameter uncertainties simultaneously in the system matrix and the input matrix. Note that only the article of Chou et al. [1] considered both structured and unstructured parameter uncertainties, and the sufficient conditions of Chou et al. [1] are the generalized versions of the results given by Lin et al. [7].

The purpose of this paper is to propose a new approach to investigate the robust controllability problems of linear descriptor systems with both structured (elemental) and unstructured (norm-bounded) parameter uncertainties in the system matrix and the input matrix. The main results are presented in Section 2. In Section 3, we extend the results given in Section 2 to study the robust controllability problems of a class of linear descriptor systems with structured uncertainties in the structure information matrix as well as with both structured and unstructured parameter uncertainties in the system matrix and the input matrix simultaneously. Three numerical examples are given in Section 4 to illustrate the applications of the proposed sufficient conditions, and to make some comparisons between the proposed sufficient conditions and those of Lin et al. [8,9] and Chou et al. [1]. Finally, Section 5 offers some conclusions.

## 2. Controllability robustness

Consider the linear continuous-time uncertain descriptor system described by

$$E\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} \alpha_i A_i x(t) + \widetilde{A}x(t) + Bu(t) + \sum_{i=1}^{m} \alpha_i B_i u(t) + \widetilde{B}u(t), \tag{1}$$

or the linear discrete-time uncertain descriptor system described by

$$Ex(k+1) = Ax(k) + \sum_{i=1}^{m} \alpha_i A_i x(k) + \widetilde{A}x(k) + Bu(k) + \sum_{i=1}^{m} \alpha_i B_i u(k) + \widetilde{B}u(k), \tag{2}$$

where $E \in R^{n\times n}$ is the structure information matrix, $A \in R^{n\times n}$ is the system matrix, and $B \in R^{n\times q}$ is the input matrix; $x(t)$ and $x(k)$ are the $n \times 1$ state vectors; $u(t)$ and $u(k)$ are the $q \times 1$ input vectors; $\alpha_i$ $(i = 1, 2, \dots, m)$ are the uncertain parameters; $A_i$ and $B_i$ $(i = 1, 2, \dots, m)$ are the given $n \times n$ and $n \times q$, respectively, constant matrices which are prescribed a prior to denote the linearly dependent information on uncertain parameters $\alpha_i$; the unstructured uncertain matrices $\widetilde{A}$ and $\widetilde{B}$ are assumed to be bounded, i.e.,

$$\|\widetilde{A}\| \leqslant \beta_1 \tag{3}$$

and

$$\|\widetilde{B}\| \leqslant \beta_2, \tag{4}$$

where $\beta_1$ and $\beta_2$ are nonnegative real constant numbers, and $\|\cdot\|$ denotes any matrix norm. Here the matrix $E$ may be a singular matrix with $\operatorname{rank}(E) \leqslant n$. In many applications, the matrix $E$ is a structure information matrix rather than a parameter matrix, i.e., the elements of $E$ contain only structure information regarding the problem considered.

In this paper, the linear nominal system $(E, A, B)$ is assumed to be regular and controllable. Due to inevitable uncertainties, the linear nominal system $(E, A, B)$ is perturbed into the linear uncertain system $(E, A + \Delta A, B + \Delta B)$, where $\Delta A = \sum_{i=1}^{m} \alpha_i A_i + \widetilde{A}$ and $\Delta B = \sum_{i=1}^{m} \alpha_i B_i + \widetilde{B}$. Our problem is to determine the conditions such that the linear uncertain system $(E, A + \Delta A, B + \Delta B)$ is still regular and controllable. Although only the controllability problems are considered, the corresponding results for the dual observability robustness problems are straightforward extensions and are omitted.

Before we investigate the robust properties of regularity and controllability of the linear uncertain system $(E, A + \Delta A, B + \Delta B)$, the following definitions and lemmas need to be introduced first.

**Definition 1** [15]. The system $(E, A, B)$ is called completely controllable (C-controllable), if for any $t_1 > 0$ (or $k_1 > 0$), $x(0) \in R^n$ and $w \in R^n$, there exists a control input $u(t)$ (or $u(k)$) such that $x(t_1) = w$ (or $x(k_1) = w$).

**Definition 2** [15]. The system $(E, A, B)$ is called R-controllable, if it is controllable in the reachable set.

**Definition 3** [10]. The system $(E, A, B)$ is called impulse controllable (I-controllable), if there is a state feedback $u(t) = Kx(t)$ (or $u(k) = Kx(k)$) such that the closed-loop system $(E, A + BK)$ is impulse-free.

**Definition 4** [10]. The system $(E, A, B)$ is called strongly controllable (S-controllable), if it is both R-controllable and I-controllable.

**Definition 5** [4]. The measure of a matrix $W \in C^{n\times n}$ is defined as

$$\mu(W) \equiv \lim_{\theta\to 0} \frac{(\|I + \theta W\| - 1)}{\theta},$$

where $\|\cdot\|$ is the induced matrix norm on $C^{n\times n}$.

**Lemma 1** [15]. *The system* $(E, A, B)$ *is regular if and only if* $\operatorname{rank}[E_n \ \ E_d] = n^2$, *where* $E_n \in R^{n^2\times n}$ *and* $E_d \in R^{n^2\times n^2}$ *are given by*

$$E_n = \begin{bmatrix} E \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \quad \text{and} \quad E_d = \begin{bmatrix} A & & & & \\ E & A & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & E & A \end{bmatrix}. \tag{5}$$

**Lemma 2** [7]. *Suppose that the system* $(E, A, B)$ *is regular. The system* $(E, A, B)$ *is I-controllable if and only if*

$$\operatorname{rank}[AS_E \quad E \quad B] = n, \tag{6}$$

*where* $S_E \in R^{n\times(n-r)}$ *is the maximum right annihilator matrix of* $E$, *in which* $r = \operatorname{rank}[E]$.

**Lemma 3** [3]. *Suppose that the system* $(E, A, B)$ *is regular. The system* $(E, A, B)$ *is R-controllable if and only if* $\operatorname{rank}[E_d \ \ E_b] = n^2$, *where* $E_d \in R^{n^2\times n^2}$ *is given in Eq.* (5) *and* $E_b = \operatorname{diag}\{B, B, \ldots, B\} \in R^{n^2\times nq}$.

**Lemma 4** [2]. *Suppose that the system* $(E, A, B)$ *is regular. The system* $(E, A, B)$ *is C-controllable if and only if it is R-controllable and* $\operatorname{rank}[E \ \ B] = n$.

**Lemma 5** [4]. *The matrix measures of the matrices* $W$ *and* $V$, $\mu(W)$ *and* $\mu(V)$, *are well defined for any norm and have the following properties*:

(i) $\mu(\pm I) = \pm 1$, *for the identity matrix* $I$;
(ii) $-\|W\| \leqslant -\mu(-W) \leqslant \operatorname{Re}(\lambda(W)) \leqslant \mu(W) \leqslant \|W\|$, *for any norm* $\|\cdot\|$ *and any matrix* $W \in C^{n\times n}$;
(iii) $\mu(W + V) \leqslant \mu(W) + \mu(V)$, *for any two matrices* $W, V \in C^{n\times n}$;
(iv) $\mu(\gamma W) = \gamma\mu(W)$, *for any matrix* $W \in C^{n\times n}$ *and any nonnegative real number* $\gamma$;

*where* $\lambda(W)$ *denotes any eigenvalue of* $W$, *and* $\operatorname{Re}(\lambda(W))$ *denotes the real part of* $\lambda(W)$.

**Lemma 6.** *For any* $\gamma < 0$ *and any matrix* $W \in C^{n\times n}$, $\mu(\gamma W) = -\gamma\mu(-W)$.

**Proof.** From the property (iv) in Lemma 5, this lemma can be immediately obtained. □

**Lemma 7.** *Let* $W \in C^{n\times n}$. *If* $\mu(-W) < 1$, *then* $\det(I + W) \neq 0$.

**Proof.** Since $\mu(-W) < 1$, then, from the property (ii) in Lemma 5, we have $\operatorname{Re}(\lambda(W)) \geqslant -\mu(-W) > -1$. This implies that $\lambda(W) \neq -1$. Thus, we can get that $\det(I + W) \neq 0$. □

Now, let the singular value decompositions of $R_0 = [E_n \ \ E_d]$, $N_0 = [AS_E \ \ E \ \ B]$, $Q_0 = [E_d \ \ E_b]$ and $M_0 = [E \ \ B]$ be, respectively,

$$R_0 = U[S \quad O_{n^2\times n}]V^{\mathrm{H}}, \tag{7}$$

$$N_0 = U_{\mathrm{I}}[S_{\mathrm{I}} \quad O_{n\times(n-r+q)}]V_{\mathrm{I}}^{\mathrm{H}}, \tag{8}$$

$$Q_0 = U_{\mathrm{R}}[S_{\mathrm{R}} \quad O_{n^2\times nq}]V_{\mathrm{R}}^{\mathrm{H}}, \tag{9}$$

and

$$M_0 = U_{\mathrm{C}}[S_{\mathrm{C}} \quad O_{n\times q}]V_{\mathrm{C}}^{\mathrm{H}}, \tag{10}$$

where $U \in R^{n^2\times n^2}$ and $V \in R^{(n^2+n)\times(n^2+n)}$ are the unitary matrices, $S = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{n^2}\}$, and $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n^2} > 0$ are the singular values of $R_0$; $U_{\mathrm{I}} \in R^{n\times n}$ and $V_{\mathrm{I}} \in R^{(2n-r+q)\times(2n-r+q)}$ are the unitary matrices, $r = \mathrm{rank}(E)$, $S_{\mathrm{I}} = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, and $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$ are the singular values of $N_0$; $U_{\mathrm{R}} \in R^{n^2\times n^2}$ and $V_{\mathrm{R}} \in R^{(n^2+nq)\times(n^2+nq)}$ are the unitary matrices, $S_{\mathrm{R}} = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_{n^2}\}$, and $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_{n^2} > 0$ are the singular values of $Q_0$; $U_{\mathrm{C}} \in R^{n\times n}$ and $V_{\mathrm{C}} \in R^{(n+q)\times(n+q)}$ are the unitary matrices, $S_{\mathrm{C}} = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, and $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n > 0$ are the singular values of $M_0$; $V^{\mathrm{H}}$, $V_{\mathrm{I}}^{\mathrm{H}}$, $V_{\mathrm{R}}^{\mathrm{H}}$ and $V_{\mathrm{C}}^{\mathrm{H}}$ denote, respectively, the complex-conjugate transposes of the matrices $V$, $V_{\mathrm{I}}$, $V_{\mathrm{R}}$ and $V_{\mathrm{C}}$.

In what follows, with the preceding definitions and lemmas, we present some sufficient conditions for ensuring that the linear uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ remains regular and controllable.

**Theorem 1.** *Suppose that the linear nominal descriptor system* $(E, A, B)$ *is regular and I-controllable. The linear uncertain descriptor system* $(E, A + \Delta A, B + \Delta B)$ *is still regular and I-controllable, if the following inequalities simultaneously hold*

$$\sum_{i=1}^{m} \alpha_i \varphi_i + \beta_1 \|S^{-1}U^{\mathrm{H}}\| \left\| V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}} \right\| < 1 \tag{11a}$$

*and*

$$\sum_{i=1}^{m} \alpha_i \phi_i + \beta_1 \|S_{\mathrm{I}}^{-1}U_{\mathrm{I}}^{\mathrm{H}}\| \|S_E\| \left\| V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} \right\| + \beta_2 \|S_{\mathrm{I}}^{-1}U_{\mathrm{I}}^{\mathrm{H}}\| \left\| V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} \right\| < 1, \tag{11b}$$

*where* $I_{n^2}$ *and* $I_n$ *denote, respectively, the* $n^2 \times n^2$ *and* $n \times n$ *identity matrices*;

$$\varphi_i = \begin{cases} \mu\left(-S^{-1}U^{\mathrm{H}}\widetilde{R}_i V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right) & \text{for } \alpha_i \geqslant 0, \\ -\mu\left(S^{-1}U^{\mathrm{H}}\widetilde{R}_i V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right) & \text{for } \alpha_i < 0; \end{cases}$$

$$\phi_i = \begin{cases} \mu(-S_{\mathrm{I}}^{-1}U_{\mathrm{I}}^{\mathrm{H}}N_i V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}) & \text{for } \alpha_i \geqslant 0, \\ -\mu\left(S_{\mathrm{I}}^{-1}U_{\mathrm{I}}^{\mathrm{H}}N_i V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\right) & \text{for } \alpha_i < 0; \end{cases}$$

$$\widetilde{R}_i = [O_{n^2\times n} \quad R_i] \in R^{n^2\times(n^2+n)};$$

$$R_i = \mathrm{diag}\{A_i, \ldots, A_i\} \in R^{n^2\times n^2};$$

$$N_i = [A_i S_E \quad O_{n\times n} \quad B_i] \in R^{n\times(2n-r+q)};$$

*the matrices* $S$, $U$, $V$, $S_{\mathrm{I}}$, $U_{\mathrm{I}}$ *and* $V_{\mathrm{I}}$ *are defined in Eqs.* (7) *and* (8), *respectively.*

**Proof.** Firstly, we show the regularity. Since the nominal system $(E, A, B)$ is regular, then, from Lemma 1, we can get that the matrix $R_0 = [E_n \ \ E_d] \in R^{n^2\times(n^2+n)}$ has full row rank (i.e.,

$\mathrm{rank}(R_0) = n^2$). With the uncertain matrices $A + \Delta A$ and $B + \Delta B$, the uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ is regular if and only if

$$\widetilde{R} = R_0 + \sum_{i=1}^{m} \alpha_i \widetilde{R}_i + \widetilde{F}, \tag{12}$$

has full row rank, where $\widetilde{R}_i = [O_{n^2\times n} \;\; R_i] \in R^{n^2\times(n^2+n)}$, $\widetilde{F} = [O_{n^2\times n} \;\; F] \in R^{n^2\times(n^2+n)}$, $R_i = \mathrm{diag}\{A_i, \ldots, A_i\} \in R^{n^2\times n^2}$, and $F = \mathrm{diag}\{\widetilde{A}, \ldots, \widetilde{A}\} \in R^{n^2\times n^2}$.

It is known that

$$\mathrm{rank}(\widetilde{R}) = \mathrm{rank}(S^{-1}U^{\mathrm{H}}\widetilde{R}V). \tag{13}$$

Thus, instead of $\mathrm{rank}(\widetilde{R})$, we can discuss the rank of

$$[I_{n^2}, O_{n^2\times n}] + \sum_{i=1}^{m} \alpha_i \widehat{R}_i + \widehat{F}, \tag{14}$$

where $\widehat{R}_i = S^{-1}U^{\mathrm{H}}\widetilde{R}_i V$ and $\widehat{F} = S^{-1}U^{\mathrm{H}}\widetilde{F}V$, for $i = 1, 2, \ldots, m$. Since a matrix has at least rank $n^2$ if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient condition for the matrix in Eq. (14) to have rank $n^2$ is the nonsingularity of

$$L = I_{n^2} + \sum_{i=1}^{m} \alpha_i \overline{R}_i + \overline{F}, \tag{15}$$

where $\overline{R}_i = S^{-1}U^{\mathrm{H}}\widetilde{R}_i V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}$ (for $i = 1, 2, \ldots, m$), and $\overline{F} = S^{-1}U^{\mathrm{H}}\widetilde{F}V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}$.

Using the properties in Lemmas 5 and 6, and from (3) and (11a), we get

$$\begin{aligned}
&\mu\left(-\sum_{i=1}^{m} \alpha_i \overline{R}_i - \overline{F}\right) \\
&\quad = \mu\left(-\sum_{i=1}^{m} \alpha_i S^{-1}U^{\mathrm{H}}\widetilde{R}_i V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}} - S^{-1}U^{\mathrm{H}}\widetilde{F}V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right) \\
&\quad \leqslant \mu\left(-\sum_{i=1}^{m} \alpha_i S^{-1}U^{\mathrm{H}}\widetilde{R}_i V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right) + \|S^{-1}U^{\mathrm{H}}\|\|\widetilde{F}\|\left\|V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right\| \\
&\quad \leqslant \sum_{i=1}^{m} \mu\big(-\alpha_i S^{-1}U^{\mathrm{H}}\widetilde{R}_i V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\big) + \|S^{-1}U^{\mathrm{H}}\|\|\widetilde{F}\|\left\|V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right\| \\
&\quad = \sum_{i=1}^{m} \alpha_i \varphi_i + \|S^{-1}U^{\mathrm{H}}\|\|\widetilde{F}\|\left\|V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right\| \\
&\quad \leqslant \sum_{i=1}^{m} \alpha_i \varphi_i + \beta_1\|S^{-1}U^{\mathrm{H}}\|\left\|V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\right\| \\
&\quad < 1.
\end{aligned} \tag{16}$$

From Lemma 7, we have that

$$\det\left(I_{n^2} + \sum_{i=1}^{m} \alpha_i \overline{R}_i + \overline{F}\right) \neq 0. \tag{17}$$

This implies that the matrix $\widetilde{R}$ has full row rank. Thus, from Lemma 1, the regularity of the uncertain descriptor system $(E, A+\Delta A, B+\Delta B)$ is ensured.

Next, we show the I-controllable. Since the nominal system $(E, A, B)$ is I-controllable, then, from Lemma 2, we can get that the matrix $N_0 = [AS_E \;\; E \;\; B]$ has full row rank (i.e., $\operatorname{rank}(N_0) = n$). With the uncertain matrices $A+\Delta A$ and $B+\Delta B$, the uncertain descriptor system $(E, A+\Delta A, B+\Delta B)$ is I-controllable if and only if

$$N = N_0 + \sum_{i=1}^{m} \alpha_i N_i + H_1 + H_2, \tag{18}$$

has full row rank, where $r = \operatorname{rank}(E)$, $N_i = [A_i S_E \;\; O_{n\times n} \;\; B_i] \in R^{n\times(2n-r+q)}$, $H_1 = [\widetilde{A} S_E \;\; O_{n\times n} \;\; O_{n\times q}] \in R^{n\times(2n-r+q)}$, and $H_2 = [O_{n\times(n-r)} \;\; O_{n\times n} \;\; \widetilde{B}] \in R^{n\times(2n-r+q)}$.

It is known that

$$\operatorname{rank}(N) = \operatorname{rank}(S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} N V_{\mathrm{I}}). \tag{19}$$

Thus, instead of $\operatorname{rank}(N)$, we can discuss the rank of

$$[I_n, O_{n\times(n-r+q)}] + \sum_{i=1}^{m} \alpha_i \widehat{N}_i + \widehat{H}_1 + \widehat{H}_2, \tag{20}$$

where $\widehat{N}_i = S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} N_i V_{\mathrm{I}}$, $\widehat{H}_1 = S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} H_1 V_{\mathrm{I}}$ and $\widehat{H}_2 = S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} H_2 V_{\mathrm{I}}$, for $i = 1, 2, \ldots, m$. Since a matrix has at least rank $n$ if it has at least one nonsingular $n \times n$ submatrix, a sufficient condition for the matrix in Eq. (20) to have rank $n$ is the nonsingularity of

$$L_{\mathrm{I}} = I_n + \sum_{i=1}^{m} \alpha_i \overline{N}_i + \overline{H}_1 + \overline{H}_2, \tag{21}$$

where $\overline{N}_i = S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} N_i V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}$ (for $i = 1, 2, \ldots, m$), $\overline{H}_1 = S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} H_1 V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}$ and $\overline{H}_2 = S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} H_2 V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}$.

Using the properties in Lemmas 5 and 6, and from (3), (4) and (11b), we have

$$\begin{aligned}
&\mu\left(-\sum_{i=1}^{m} \alpha_i \overline{N}_i - \overline{H}_1 - \overline{H}_2\right) \\
&\quad = \mu\Bigg(-\sum_{i=1}^{m} \alpha_i S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} N_i V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} - S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} H_1 V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} \\
&\qquad - S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} H_2 V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\Bigg) \\
&\quad \leqslant \mu\left(-\sum_{i=1}^{m} \alpha_i S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} N_i V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\right) \\
&\qquad + \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \|H_1\| \left\| V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} \right\| + \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \|H_2\| \left\| V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} \right\| \\
&\quad \leqslant \sum_{i=1}^{m} \mu\big(-\alpha_i S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} N_i V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big) \\
&\qquad + \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \|H_1\| \left\| V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} \right\| + \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \|H_2\| \left\| V_{\mathrm{I}} [I_n, O_{n\times(n-r+q)}]^{\mathrm{T}} \right\|
\end{aligned}$$

$$\begin{aligned}&\leqslant \sum_{i=1}^{m} \mu\big(-\alpha_i S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}} N_i V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big)\\ &\quad + \beta_1 \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \|S_E\| \big\|V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big\| + \beta_2 \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \big\|V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big\|\\ &= \sum_{i=1}^{m} \alpha_i \phi_i + \beta_1 \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \|S_E\| \big\|V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big\|\\ &\quad + \beta_2 \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \big\|V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big\|\\ &< 1. \end{aligned} \tag{22}$$

From Lemma 7, we have that

$$\det\left(I_n + \sum_{i=1}^{m} \alpha_i \overline{N}_i + \overline{H}_1 + \overline{H}_2\right) \neq 0. \tag{23}$$

This implies that the matrix $N$ has full row rank. Hence, from the results mentioned above and Lemma 2, the I-controllability of the uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ is ensured. □

**Theorem 2.** *Suppose that the linear nominal descriptor system* $(E, A, B)$ *is regular and R-controllable. The linear uncertain descriptor system* $(E, A + \Delta A, B + \Delta B)$ *is still regular and R-controllable, if the following inequalities simultaneously hold*

$$\sum_{i=1}^{m} \alpha_i \varphi_i + \beta_1 \|S^{-1} U^{\mathrm{H}}\| \big\|V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\big\| < 1 \tag{24a}$$

*and*

$$\sum_{i=1}^{m} \alpha_i \theta_i + (\beta_1 + \beta_2) \big\|S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}}\big\| \big\|V_{\mathrm{R}}[I_{n^2}, O_{n^2\times nq}]^{\mathrm{T}}\big\| < 1, \tag{24b}$$

*where*

$$\theta_i = \begin{cases} \mu\big(-S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}}[I_{n^2}, O_{n^2\times nq}]^{\mathrm{T}}\big) & \text{for } \alpha_i \geqslant 0,\\ -\mu\big(S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}}[I_{n^2}, O_{n^2\times nq}]^{\mathrm{T}}\big) & \text{for } \alpha_i < 0; \end{cases}$$

$$Q_i = \begin{bmatrix} A_i & & & & B_i & & & \\ & A_i & & & & B_i & & \\ & & \ddots & & & & \ddots & \\ & & & A_i & & & & B_i \end{bmatrix} \in R^{n^2\times(n^2+nq)};$$

$\varphi_i$ $(i = 1, 2, \ldots, m)$ *are given in Theorem* 1; *the matrices* $S, U, V, S_{\mathrm{R}}, U_{\mathrm{R}}$ *and* $V_{\mathrm{R}}$ *are defined in Eqs.* (7) *and* (9), *respectively.*

**Proof.** Firstly, following the same proof procedure given in Theorem 1, we can prove that the sufficient condition (24a) ensures the uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ to be regular. Next, we show the R-controllability.

Since the nominal system $(E, A, B)$ is R-controllable, then, from Lemma 3, we have that the matrix $Q_0 = [E_d \;\; E_b]$ has full row rank (i.e., $\mathrm{rank}(Q_0) = n^2$). With the uncertain matrices $A +$

$\Delta A$ and $B + \Delta B$, the linear uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ is R-controllable if and only if

$$Q = Q_0 + \sum_{i=1}^{m} \alpha_i Q_i + G_1 + G_2, \tag{25}$$

has full row rank, where

$$Q_i = \begin{bmatrix} A_i & & & & B_i & & & \\ & A_i & & & & B_i & & \\ & & \ddots & & & & \ddots & \\ & & & A_i & & & & B_i \end{bmatrix} \in R^{n^2 \times (n^2+nq)},$$

$$G_1 = [F \quad O_{n^2 \times nq}] \in R^{n^2 \times (n^2+nq)},$$

and

$$G_2 = [O_{n^2 \times n^2} \quad \operatorname{diag}\{\widetilde{B}, \ldots, \widetilde{B}\}] \in R^{n^2 \times (n^2+nq)}.$$

It is known that

$$\operatorname{rank}(Q) = \operatorname{rank}(S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q V_{\mathrm{R}}). \tag{26}$$

Thus, instead of $\operatorname{rank}(Q)$, we can discuss the rank of

$$[I_{n^2}, O_{n^2 \times nq}] + \sum_{i=1}^{m} \alpha_i \widehat{Q}_i + \widehat{G}_1 + \widehat{G}_2, \tag{27}$$

where $\widehat{Q}_i = S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}}$, $\widehat{G}_1 = S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} G_1 V_{\mathrm{R}}$ and $\widehat{G}_2 = S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} G_2 V_{\mathrm{R}}$, for $i = 1, 2, \ldots, m$. Since a matrix has at least rank $n^2$ if it has at least one nonsingular $n^2 \times n^2$ submatrix, a sufficient condition for the matrix in Eq. (27) to have rank $n^2$ is the nonsingularity of

$$L_{\mathrm{R}} = I_{n^2} + \sum_{i=1}^{m} \alpha_i \overline{Q}_i + \overline{G}_1 + \overline{G}_2, \tag{28}$$

where $\overline{Q}_i = S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}}$ (for $i = 1, 2, \ldots, m$), $\overline{G}_1 = S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} G_1 V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}}$ and $\overline{G}_2 = S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} G_2 V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}}$.

Applying the properties in Lemmas 5 and 6, and from (3), (4) and (24b), we can get

$$\begin{aligned}
&\mu\left(-\sum_{i=1}^{m} \alpha_i \overline{Q}_i - \overline{G}_1 - \overline{G}_2\right) \\
&\quad = \mu\Bigg( -\sum_{i=1}^{m} \alpha_i S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} - S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} G_1 V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \\
&\qquad - S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} G_2 V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \Bigg) \\
&\quad \leqslant \mu\left(-\sum_{i=1}^{m} \alpha_i S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}}\right) + \left\| S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} \right\| \|G_1\| \left\| V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \right\|
\end{aligned}$$

$$
\begin{aligned}
&+ \left\| S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} \right\| \| G_2 \| \left\| V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \right\| \\
\leqslant & \sum_{i=1}^{m} \mu\big( - \alpha_i S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \big) + \left\| S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} \right\| \| G_1 \| \left\| V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \right\| \\
&+ \left\| S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} \right\| \| G_2 \| \left\| V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \right\| \\
\leqslant & \sum_{i=1}^{m} \mu\big( - \alpha_i S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} Q_i V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \big) + \beta_1 \left\| S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} \right\| \left\| V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \right\| \\
&+ \beta_2 \left\| S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} \right\| \left\| V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \right\| \\
= & \sum_{i=1}^{m} \alpha_i \theta_i + (\beta_1 + \beta_2) \left\| S_{\mathrm{R}}^{-1} U_{\mathrm{R}}^{\mathrm{H}} \right\| \left\| V_{\mathrm{R}} [I_{n^2}, O_{n^2 \times nq}]^{\mathrm{T}} \right\| \\
< & \ 1.
\end{aligned} \tag{29}
$$

From Lemma 7, we have that

$$
\det \left( I_{n^2} + \sum_{i=1}^{m} \alpha_i \overline{Q}_i + \overline{G}_1 + \overline{G}_2 \right) \neq 0. \tag{30}
$$

This implies that the matrix $Q$ has full row rank. Thus, from Lemma 3 and the results mentioned above, the proof is completed. □

**Theorem 3.** *Suppose that the linear nominal descriptor system* $(E, A, B)$ *is C-controllable. The linear uncertain descriptor system* $(E, A + \Delta A, B + \Delta B)$ *is still C-controllable, if the inequalities in* (24a) *and* (24b) *and the following inequality simultaneously hold*:

$$
\sum_{i=1}^{m} \alpha_i \sigma_i + \beta_2 \left\| S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \right\| \left\| V_{\mathrm{C}} [I_n, O_{n \times q}]^{\mathrm{T}} \right\| < 1, \tag{31}
$$

*where*

$$
\sigma_i = \begin{cases} \mu\big( - S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}} [I_n, O_{n \times q}]^{\mathrm{T}} \big) & \text{for } \alpha_i \geqslant 0, \\ -\mu\big( S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}} [I_n, O_{n \times q}]^{\mathrm{T}} \big) & \text{for } \alpha_i < 0; \end{cases}
$$

$$
M_i = [O_{n \times n} \quad B_i] \in R^{n \times (n+q)};
$$

*the matrices* $S_{\mathrm{C}}$, $U_{\mathrm{C}}$ *and* $V_{\mathrm{C}}$ *are defined in Eq.* (10).

**Proof.** Since the nominal system $(E, A, B)$ is C-controllable, then, from Lemma 4, we can obtain that the matrix $M_0 = [E \quad B]$ has full row rank (i.e., $\mathrm{rank}(M_0) = n$). With the uncertain matrices $A + \Delta A$ and $B + \Delta B$, the uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ is C-controllable if and only if

$$
M = M_0 + \sum_{i=1}^{m} \alpha_i M_i + W, \tag{32}
$$

has full row rank, where $M_i = [O_{n \times n} \quad B_i] \in R^{n \times (n+q)}$, and $W = [O_{n \times n} \quad \widetilde{B}] \in R^{n \times (n+q)}$.

It is known that

$$
\mathrm{rank}(M) = \mathrm{rank}(S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M V_{\mathrm{C}}). \tag{33}
$$

Thus, instead of $\operatorname{rank}(M)$, we can discuss the rank of

$$[I_n, O_{n\times q}] + \sum_{i=1}^{m} \alpha_i \widehat{M}_i + \widehat{W}, \tag{34}$$

where $\widehat{M}_i = S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}}$ and $\widehat{W} = S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} W V_{\mathrm{C}}$, for $i = 1, 2, \ldots, m$. Since a matrix has at least rank $n$ if it has at least one nonsingular $n \times n$ submatrix, a sufficient condition for the matrix in Eq. (34) to have rank $n$ is the nonsingularity of

$$L_{\mathrm{C}} = I_n + \sum_{i=1}^{m} \alpha_i \overline{M}_i + \overline{W}, \tag{35}$$

where $\overline{M}_i = S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}$ (for $i = 1, 2, \ldots, m$), and $\overline{W} = S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} W V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}$.

Adopting the properties in Lemmas 5 and 6, and from (4) and (31), we obtain

$$\begin{aligned}
&\mu\left(-\sum_{i=1}^{m} \alpha_i \overline{M}_i - \overline{W}\right) \\
&\quad = \mu\left(-\sum_{i=1}^{m} \alpha_i S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}} - S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} W V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}\right) \\
&\quad \leqslant \mu\left(-\sum_{i=1}^{m} \alpha_i S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}\right) + \left\| S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \right\| \|W\| \left\| V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}} \right\| \\
&\quad \leqslant \sum_{i=1}^{m} \mu\left(-\alpha_i S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}\right) + \left\| S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \right\| \|W\| \left\| V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}} \right\| \\
&\quad \leqslant \sum_{i=1}^{m} \mu\left(-\alpha_i S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} M_i V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}\right) + \beta_2 \left\| S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \right\| \left\| V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}} \right\| \\
&\quad = \sum_{i=1}^{m} \alpha_i \sigma_i + \beta_2 \left\| S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \right\| \left\| V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}} \right\| \\
&\quad < 1.
\end{aligned} \tag{36}$$

From Lemma 7, we get that

$$\det\left(I_n + \sum_{i=1}^{m} \alpha_i \overline{M}_i + \overline{W}\right) \neq 0. \tag{37}$$

This implies that the matrix $M$ has full row rank. Thus, from Lemma 4 and the results mentioned above, we can conclude that the uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ is C-controllable, if the inequalities (24a), (24b) and (31) are simultaneously satisfied. □

**Remark 1.** From Definition 4, the sufficient conditions that ensure the S-controllability of the uncertain descriptor system $(E, A + \Delta A, B + \Delta B)$ can be immediately obtained from Theorems 1 and 2.

**Remark 2.** Due to the relationships between each observability and its corresponding controllability, the results for robust observability are straightforward.

**Remark 3.** The proposed sufficient conditions in (11a), (11b), (24a), (24b) and (31) can give the explicit relationship of the bounds on $\alpha_i$ $(i = 1, 2, \ldots, m)$, $\beta_1$ and $\beta_2$ for preserving both regularity and controllability. In addition, the bounds, that are obtained by using these proposed sufficient conditions, on $\alpha_i$ are not necessarily symmetric with respect to the origin of the parameter space regarding $\alpha_i$.

## 3. Descriptor systems with structure information uncertainties and parameter uncertainties

Sometimes we have to deal with the systems simultaneously having both structure information uncertainties and parameter uncertainties. Therefore, in this section, we consider a class of linear continuous-time descriptor system with both structure information uncertainties and parameter uncertainties described by

$$\left(E + \sum_{i=1}^{m} \alpha_i E_i\right) \dot{x}(t) = Ax(t) + \sum_{i=1}^{m} \alpha_i A_i x(t) + \widetilde{A}x(t) + Bu(t) + \sum_{i=1}^{m} \alpha_i B_i u(t) + \widetilde{B}u(t), \tag{38}$$

or a class of linear discrete-time descriptor system with both structure information uncertainties and parameter uncertainties described by

$$\left(E + \sum_{i=1}^{m} \alpha_i E_i\right) x(k+1) = Ax(k) + \sum_{i=1}^{m} \alpha_i A_i x(k) + \widetilde{A}x(k) + Bu(k) + \sum_{i=1}^{m} \alpha_i B_i u(k) + \widetilde{B}u(k). \tag{39}$$

Here we use $(E + \Delta E, A + \Delta A, B + \Delta B)$ to denote the linear uncertain descriptor system in Eq. (38) or (39).

Following the same proof procedures given in Theorems 1–3, we can get the following corollaries to ensure that the uncertain descriptor system $(E + \Delta E, A + \Delta A, B + \Delta B)$ remains regular, I-controllable, R-controllable and C-controllable.

**Corollary 1.** *Suppose that the linear nominal descriptor system $(E, A, B)$ is regular and I-controllable. The linear uncertain descriptor system $(E + \Delta E, A + \Delta A, B + \Delta B)$ is still regular and I-controllable, if the following inequalities simultaneously hold:*

$$\sum_{i=1}^{m} \alpha_i \bar{\varphi}_i + \beta_1 \|S^{-1}U^{\mathrm{H}}\| \left\| V[I_{n^2}, O_{n^2 \times n}]^{\mathrm{T}} \right\| < 1 \tag{40a}$$

*and*

$$\sum_{i=1}^{m} \alpha_i \bar{\phi}_i + \beta_1 \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \|S_E\| \left\| V_{\mathrm{I}}[I_n, O_{n \times (n-r+q)}]^{\mathrm{T}} \right\| + \beta_2 \|S_{\mathrm{I}}^{-1} U_{\mathrm{I}}^{\mathrm{H}}\| \left\| V_{\mathrm{I}}[I_n, O_{n \times (n-r+q)}]^{\mathrm{T}} \right\| < 1, \tag{40b}$$

*where*

$$\bar{\varphi}_i = \begin{cases} \mu\big(-S^{-1}U^{\mathrm{H}}\widetilde{R}_i^* V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\big) & \text{for } \alpha_i \geqslant 0, \\ -\mu\big(S^{-1}U^{\mathrm{H}}\widetilde{R}_i^* V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}}\big) & \text{for } \alpha_i < 0; \end{cases}$$

$$\bar{\phi}_i = \begin{cases} \mu\big(-S_{\mathrm{I}}^{-1}U_{\mathrm{I}}^{\mathrm{H}}\widetilde{N}_i V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big) & \text{for } \alpha_i \geqslant 0, \\ -\mu\big(S_{\mathrm{I}}^{-1}U_{\mathrm{I}}^{\mathrm{H}}\widetilde{N}_i V_{\mathrm{I}}[I_n, O_{n\times(n-r+q)}]^{\mathrm{T}}\big) & \text{for } \alpha_i < 0; \end{cases}$$

$$\widetilde{R}_i^* = [E_i^* \quad R_i^* \;] \in R^{n^2\times(n^2+n)};$$

$$E_i^* = \begin{bmatrix} E_i \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \in R^{n^2\times n}; \quad R_i^* = \begin{bmatrix} A_i & & & & \\ E_i & A_i & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & E_i & A_i \end{bmatrix} \in R^{n^2\times n^2};$$

$$\widetilde{N}_i = [A_i S_E \quad E_i \quad B_i] \in R^{n\times(2n-r+q)};$$

*the matrices $S$, $U$, $V$, $S_{\mathrm{I}}$, $U_{\mathrm{I}}$ and $V_{\mathrm{I}}$ are defined in Eqs.* (7) *and* (8), *respectively.*

**Proof.** The proof procedure of Corollary 1 is similar to that of Theorem 1, hence omitted here. □

**Corollary 2.** *Suppose that the linear nominal descriptor system $(E, A, B)$ is regular and R-controllable. The linear uncertain descriptor system $(E + \Delta E, A + \Delta A, B + \Delta B)$ is still regular and R-controllable, if the following inequalities simultaneously hold*:

$$\sum_{i=1}^{m} \alpha_i \bar{\varphi}_i + \beta_1 \|S^{-1}U^{\mathrm{H}}\| \big\| V[I_{n^2}, O_{n^2\times n}]^{\mathrm{T}} \big\| < 1 \tag{41a}$$

*and*

$$\sum_{i=1}^{m} \alpha_i \bar{\theta}_i + (\beta_1 + \beta_2) \big\| S_{\mathrm{R}}^{-1}U_{\mathrm{R}}^{\mathrm{H}} \big\| \big\| V_{\mathrm{R}}[I_{n^2}, O_{n^2\times nq}]^{\mathrm{T}} \big\| < 1, \tag{41b}$$

*where*

$$\bar{\theta}_i = \begin{cases} \mu(-S_{\mathrm{R}}^{-1}U_{\mathrm{R}}^{\mathrm{H}}\widetilde{Q}_i V_{\mathrm{R}}[I_{n^2}, O_{n^2\times nq}]^{\mathrm{T}}) & \text{for } \alpha_i \geqslant 0, \\ -\mu(S_{\mathrm{R}}^{-1}U_{\mathrm{R}}^{\mathrm{H}}\widetilde{Q}_i V_{\mathrm{R}}[I_{n^2}, O_{n^2\times nq}]^{\mathrm{T}}) & \text{for } \alpha_i < 0; \end{cases}$$

$$\widetilde{Q}_i = \begin{bmatrix} A_i & & & & & B_i & & & \\ E_i & A_i & & & & & B_i & & \\ & \cdot & \cdot & & & & & \cdot & \\ & & \cdot & \cdot & & & & & \cdot \\ & & & \cdot & \cdot & & & & \cdot \\ & & & E_i & A_i & & & & B_i \end{bmatrix} \in R^{n^2\times(n^2+nq)};$$

$\bar{\varphi}_i$ $(i = 1, 2, \ldots, m)$ *are given in Corollary* 1; *the matrices $S$, $U$, $V$, $S_{\mathrm{R}}$, $U_{\mathrm{R}}$ and $V_{\mathrm{R}}$ are defined in Eqs.* (7) *and* (9), *respectively.*

**Proof.** The proof procedure of Corollary 2 is similar to that of Theorem 2, hence omitted here. □

**Corollary 3.** *Suppose that the linear nominal descriptor system* $(E, A, B)$ *is C-controllable. The linear uncertain descriptor system* $(E + \Delta E, A + \Delta A, B + \Delta B)$ *is still C-controllable, if the inequalities in* (41a) *and* (41b) *and the following inequality simultaneously hold*:

$$\sum_{i=1}^{m} \alpha_i \bar{\sigma}_i + \beta_2 \left\| S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \right\| \left\| V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}} \right\| < 1, \tag{42}$$

*where*

$$\bar{\sigma}_i = \begin{cases} \mu(-S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \widetilde{M}_i V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}) & \text{for } \alpha_i \geqslant 0, \\ -\mu(S_{\mathrm{C}}^{-1} U_{\mathrm{C}}^{\mathrm{H}} \widetilde{M}_i V_{\mathrm{C}} [I_n, O_{n\times q}]^{\mathrm{T}}) & \text{for } \alpha_i < 0; \end{cases}$$
$$\widetilde{M}_i = [E_i \quad B_i] \in R^{n\times(n+q)};$$

*the matrices* $S_{\mathrm{C}}$, $U_{\mathrm{C}}$ *and* $V_{\mathrm{C}}$ *are defined in Eq.* (10).

**Proof.** The proof procedure of Corollary 3 is similar to that of Theorem 3, hence omitted here. □

## 4. Illustrative examples

In this section, three examples are given for illustrating the applications of the proposed sufficient conditions, and making some comparisons between the proposed sufficient conditions and those of Lin et al. [8,9] and Chou et al. [1].

**Example 1.** Consider the following linear continuous-time descriptor system with structured parameter uncertainties described as

$$E\dot{x}(t) = (A + \alpha_1 A_1 + \alpha_2 A_2)x(t) + (B + \alpha_1 B_1 + \alpha_2 B_2)u(t), \tag{43}$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 1 & 2 \\ 2 & -1 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & -0.3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 0 & -0.6 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$
$$\alpha_1 \in [-0.64, 0.3] \quad \text{and} \quad \alpha_2 \in [-0.28, 0.68].$$

Now, choose the same $S_E = [0 \ \ 0 \ \ 1]^{\mathrm{T}}$ as that chosen by Lin et al. [8,9]. By using the method of Lin et al. [8,9] and the software of Matlab Toolbox for the structured singular value, we can obtain $|\alpha_i(t)| \not< \mu_\Delta^{-1}(M_{\mathrm{reg}}) = 0.035$, $|\alpha_i| \not< \mu_\Delta^{-1}(M_{\mathrm{I}}) = 0.2570$, and $|\alpha_i| \not< \mu_\Delta^{-1}(M_{\mathrm{C}}) = 0.1752$, for $i = 1, 2$. Thus, we cannot reach any conclusion for guaranteeing the robust regularity, the robust I-controllability, and the robust C-controllability. That is, the sufficient conditions of Lin et al. [8,9] cannot be applied in this example.

By adopting the method of Chou et al. [1] with the 2-norm-based matrix measure to check the robust regularity, we can obtain

(i) $2.8409\alpha_1 + 0.6380\alpha_2 \leqslant 1.2861 \not< 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.68]$;
(ii) $-0.9106\alpha_1 + 0.6380\alpha_2 \leqslant 1.0166 \not< 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [0, 0.68]$;
(iii) $-0.9106\alpha_1 - 1.2292\alpha_2 \leqslant 0.9270 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $2.8409\alpha_1 - 1.2292\alpha_2 \leqslant 1.1964 \not< 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.28, 0]$.

Applying the method of Chou et al. [1] with the 2-norm-based matrix measure to check the robust I-controllability, we can get

(i) $0.5849\alpha_1 + 1.1416\alpha_2 \leqslant 0.9518 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.68]$;
(ii) $-0.5849\alpha_1 + 1.1416\alpha_2 \leqslant 1.1506 \not< 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [0, 0.68]$;
(iii) $-0.5849\alpha_1 - 1.1416\alpha_2 \leqslant 0.6940 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $0.5849\alpha_1 - 1.1416\alpha_2 \leqslant 0.4951 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.28, 0]$.

Using the method of Chou et al. [1] with the 2-norm-based matrix measure to check the robust R-controllability and the robust C-controllability, we can obtain

(i) $0.7458\alpha_1 + 0.9464\alpha_2 \leqslant 0.8673 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.68]$;
(ii) $-1.3936\alpha_1 + 0.9464\alpha_2 \leqslant 1.5354 \not< 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [0, 0.68]$;
(iii) $-1.3936\alpha_1 - 2.2273\alpha_2 \leqslant 1.5155 \not< 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $0.7458\alpha_1 - 2.2273\alpha_2 \leqslant 0.8474 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.28, 0]$.

Then, no conclusion can be made for keeping the robust regularity, the robust I-controllability, the robust R-controllability and the robust C-controllability. That is, the sufficient conditions of Chou et al. [1] cannot also be applied in this example.

Now, applying the sufficient condition (11a) with the 2-norm-based matrix measure, we have

(i) $\sum_{i=1}^{2} \alpha_i\varphi_i \leqslant 0.9983 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.68]$;
(ii) $\sum_{i=1}^{2} \alpha_i\varphi_i \leqslant 0.9325 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [0, 0.68]$;
(iii) $\sum_{i=1}^{2} \alpha_i\varphi_i \leqslant 0.9231 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i\varphi_i \leqslant 0.9889 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.28, 0]$.

Using the sufficient condition (11b) with the 2-norm-based matrix measure and $S_E = [0\ \ 0\ \ 1]^{\mathrm{T}}$ which is the same as that adopted by Lin et al. [8,9], we get

(i) $\sum_{i=1}^{2} \alpha_i\phi_i \leqslant 0.8202 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.68]$;
(ii) $\sum_{i=1}^{2} \alpha_i\phi_i \leqslant 0.9962 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [0, 0.68]$;
(iii) $\sum_{i=1}^{2} \alpha_i\phi_i \leqslant 0.5338 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i\phi_i \leqslant 0.3578 < 1$, for $\alpha_1(t) \in [0, 0.3]$ and $\alpha_2 \in [-0.28, 0]$.

Adopting the sufficient condition (24b) with the 2-norm-based matrix measure, we have

(i) $\sum_{i=1}^{2} \alpha_i\theta_i \leqslant 0.1503 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.68]$;
(ii) $\sum_{i=1}^{2} \alpha_i\theta_i \leqslant 0.5454 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [0, 0.68]$;
(iii) $\sum_{i=1}^{2} \alpha_i\theta_i \leqslant 0.7304 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i\theta_i \leqslant 0.3353 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.28, 0]$.

And applying the sufficient condition (31) with the 2-norm-based matrix measure, we obtain

(i) $\sum_{i=1}^{2} \alpha_i \sigma_i \leqslant 0.6308 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.68]$;
(ii) $\sum_{i=1}^{2} \alpha_i \sigma_i \leqslant 0.8008 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [0, 0.68]$;
(iii) $\sum_{i=1}^{2} \alpha_i \sigma_i \leqslant 0.5180 < 1$, for $\alpha_1 \in [-0.64, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \sigma_i \leqslant 0.3480 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.28, 0]$.

So, we can conclude that the linear uncertain descriptor system (43) is regular, I-controllable, R-controllable and C-controllable. From above results, it can be shown that our proposed sufficient conditions are less conservative than those of Lin et al. [8,9] and Chou et al. [1].

**Example 2.** Consider the linear continuous-time descriptor system with both structured and unstructured parameter uncertainties described by

$$E\dot{x}(t) = (A + \alpha_1 A_1 + \alpha_2 A_2 + \widetilde{A})x(t) + (B + \alpha_1 B_1 + \alpha_2 B_2 + \widetilde{B})u(t) \tag{44}$$

with $\alpha_1 \in [-0.51, 0.21]$ and $\alpha_2 \in [-0.28, 0.63]$, where $\|\widetilde{A}\| \leqslant \beta$, $\|\widetilde{B}\| \leqslant \beta$, $\beta = 0.05$, and the matrices $E$, $A$, $A_1$, $A_2$, $B$, $B_1$ and $B_2$ are the same as those given in Example 1.

By using the approach of Chou et al. [1] with the 2-norm-based matrix measure and the spectral norm to check the robust regularity, we can get

(i) $2.8409\alpha_1 + 0.6380\alpha_2 + 3.7389\beta \leqslant 1.1855 \nless 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [0, 0.63]$;
(ii) $-0.9106\alpha_1 + 0.6380\alpha_2 + 3.7389\beta \leqslant 1.05 \nless 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [0, 0.63]$;
(iii) $-0.9106\alpha_1 - 1.2292\alpha_2 + 3.7389\beta \leqslant 0.9955 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $2.8409\alpha_1 - 1.2292\alpha_2 + 3.7389\beta \leqslant 1.1277 \nless 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [-0.28, 0]$.

Adopting the approach of Chou et al. [1] with the 2-norm-based matrix measure and the spectral norm to check the robust I-controllability, we can obtain

(i) $0.5849\alpha_1 + 1.1416\alpha_2 + \beta \leqslant 0.8920 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [0, 0.63]$;
(ii) $-0.5849\alpha_1 + 1.1416\alpha_2 + \beta \leqslant 1.0675 \nless 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [0, 0.63]$;
(iii) $-0.5849\alpha_1 - 1.1416\alpha_2 + \beta \leqslant 0.6679 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $0.5849\alpha_1 - 1.1416\alpha_2 + \beta \leqslant 0.4925 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [-0.28, 0]$.

Applying the approach of Chou et al. [1] with the 2-norm-based matrix measure and the spectral norm to check the robust R-controllability and the robust C-controllability, we can get

(i) $0.7458\alpha_1 + 0.9464\alpha_2 + 2.3695\beta \leqslant 0.8713 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [0, 0.63]$;
(ii) $-1.3936\alpha_1 + 0.9464\alpha_2 + 2.3695\beta \leqslant 1.4254 \nless 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [0, 0.63]$;
(iii) $-1.3936\alpha_1 - 2.2273\alpha_2 + 2.3695\beta \leqslant 1.4528 \nless 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $0.7458\alpha_1 - 2.2273\alpha_2 + 2.3695\beta \leqslant 0.8987 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [-0.28, 0]$.

Thus, no conclusion can be made for keeping the robust regularity, the robust I-controllability, the robust R-controllability and the robust C-controllability. That is, the sufficient conditions of Chou et al. [1] cannot be applied in this example.

Now, using the sufficient condition (11a) with the 2-norm-based matrix measure and the spectral norm, we get

(i) $\sum_{i=1}^{2} \alpha_i \varphi_i + 3.8442\beta \leqslant 0.9709 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [0, 0.63]$;
(ii) $\sum_{i=1}^{2} \alpha_i \varphi_i + 3.8442\beta \leqslant 0.9810 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [0, 0.63]$;
(iii) $\sum_{i=1}^{2} \alpha_i \varphi_i + 3.8442\beta \leqslant 0.9975 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \varphi_i + 3.8442\beta \leqslant 0.9874 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [-0.28, 0]$.

Adopting the sufficient condition (11b) with the 2-norm-based matrix measure, the spectral norm, and $S_E = [0\ \ 0\ \ 1]^{\mathrm{T}}$ which is the same as that adopted by Lin et al. [8,9], we have

(i) $\sum_{i=1}^{2} \alpha_i \phi_i + 2.2258\beta \leqslant 0.8360 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [0, 0.63]$;
(ii) $\sum_{i=1}^{2} \alpha_i \phi_i + 2.2258\beta \leqslant 0.9913 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [0, 0.63]$;
(iii) $\sum_{i=1}^{2} \alpha_i \phi_i + 2.2258\beta \leqslant 0.5778 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \phi_i + 2.2258\beta \leqslant 0.4225 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [-0.28, 0]$.

Applying the sufficient condition (24b) with the 2-norm-based matrix measure and the spectral norm, we get

(i) $\sum_{i=1}^{2} \alpha_i \theta_i + 3.2174\beta \leqslant 0.2806 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [0, 0.63]$;
(ii) $\sum_{i=1}^{2} \alpha_i \theta_i + 3.2174\beta \leqslant 0.6038 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [0, 0.63]$;
(iii) $\sum_{i=1}^{2} \alpha_i \theta_i + 3.2174\beta \leqslant 0.7935 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \theta_i + 3.2174\beta \leqslant 0.4703 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [-0.28, 0]$.

And using the sufficient condition (31) with the 2-norm-based matrix measure and the spectral norm, we have

(i) $\sum_{i=1}^{2} \alpha_i \sigma_i + 1.618\beta \leqslant 0.6314 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [0, 0.63]$;
(ii) $\sum_{i=1}^{2} \alpha_i \sigma_i + 1.618\beta \leqslant 0.7814 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [0, 0.63]$;
(iii) $\sum_{i=1}^{2} \alpha_i \sigma_i + 1.618\beta \leqslant 0.5339 < 1$, for $\alpha_1 \in [-0.51, 0]$ and $\alpha_2 \in [-0.28, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \sigma_i + 1.618\beta \leqslant 0.3839 < 1$, for $\alpha_1 \in [0, 0.21]$ and $\alpha_2 \in [-0.28, 0]$.

Therefore, we can conclude that the linear uncertain descriptor system (44) is regular, I-controllable, R-controllable and C-controllable. From above results, it can be shown that our proposed sufficient conditions are less conservative than those of Chou et al. [1].

**Example 3.** Consider the following linear continuous-time descriptor system with both structure information uncertainties and parameter uncertainties described as

$$(E + \alpha_1 E_1 + \alpha_2 E_2)\dot{x}(t) = (A + \alpha_1 A_1 + \alpha_2 A_2)x(t) + (B + \alpha_1 B_1 + \alpha_2 B_2)u(t), \tag{45}$$

where

$$E_1 = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$\alpha_1 \in [-0.6, 0.3], \quad \alpha_2 \in [-0.29, 0.6],$$

and the matrices $E$, $A$, $A_1$, $A_2$, $B$, $B_1$ and $B_2$ are the same as those given in Example 1.

Now, by applying the method of Lin et al. [8,9] and the software of Matlab Toolbox for the structured singular value, we can obtain $|\alpha_i(t)| \not< \mu_\Delta^{-1}(M_{\text{reg}}) = 0.034$ and $|\alpha_i| \not< \mu_\Delta^{-1}(M_{\text{C}}) = 0.1717$, for $i = 1, 2$. Thus, we cannot reach any conclusion for guaranteeing the robust regularity and the robust C-controllability. That is, the sufficient conditions of Lin et al. [8,9] cannot be applied in this example.

Now, using the sufficient condition (41a) with the 2-norm-based matrix measure, we have

(i) $\sum_{i=1}^{2} \alpha_i \bar{\varphi}_i \leqslant 0.9550 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.6]$;
(ii) $\sum_{i=1}^{2} \alpha_i \bar{\varphi}_i \leqslant 0.8828 < 1$, for $\alpha_1 \in [-0.6, 0]$ and $\alpha_2 \in [0, 0.6]$;
(iii) $\sum_{i=1}^{2} \alpha_i \bar{\varphi}_i \leqslant 0.9206 < 1$, for $\alpha_1 \in [-0.6, 0]$ and $\alpha_2 \in [-0.29, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \bar{\varphi}_i \leqslant 0.9928 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.29, 0]$.

Adopting the sufficient condition (41b) with the 2-norm-based matrix measure, we obtain

(i) $\sum_{i=1}^{2} \alpha_i \bar{\theta}_i \leqslant 0.1435 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.6]$;
(ii) $\sum_{i=1}^{2} \alpha_i \bar{\theta}_i \leqslant 0.5155 < 1$, for $\alpha_1 \in [-0.6, 0]$ and $\alpha_2 \in [0, 0.6]$;
(iii) $\sum_{i=1}^{2} \alpha_i \bar{\theta}_i \leqslant 0.7133 < 1$, for $\alpha_1 \in [-0.6, 0]$ and $\alpha_2 \in [-0.29, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \bar{\theta}_i \leqslant 0.3413 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.29, 0]$.

And applying the sufficient condition (42) with the 2-norm-based matrix measure, we get

(i) $\sum_{i=1}^{2} \alpha_i \bar{\sigma}_i \leqslant 0.5463 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [0, 0.6]$;
(ii) $\sum_{i=1}^{2} \alpha_i \bar{\sigma}_i \leqslant 0.7420 < 1$, for $\alpha_1 \in [-0.6, 0]$ and $\alpha_2 \in [0, 0.6]$;
(iii) $\sum_{i=1}^{2} \alpha_i \bar{\sigma}_i \leqslant 0.5445 < 1$, for $\alpha_1 \in [-0.6, 0]$ and $\alpha_2 \in [-0.29, 0]$;
(iv) $\sum_{i=1}^{2} \alpha_i \bar{\sigma}_i \leqslant 0.3487 < 1$, for $\alpha_1 \in [0, 0.3]$ and $\alpha_2 \in [-0.29, 0]$.

Thus, from Corollary 3, we can conclude that the linear uncertain descriptor system (45) is regular and C-controllable. From above results, it can be shown that our proposed sufficient conditions are less conservative than those of Lin et al. [8,9].

**Remark 4.** Based on the structured singular value approach ($\mu$ analysis) and the Kronecker product, the sufficient conditions proposed by Lin et al. [8,9] are obtained by transforming the robust controllability problem into checking the nonsingularity of a class of uncertain matrices. By using the matrix measure approach, Chou et al. [1] presented some sufficient conditions to study the robust controllability problems for the linear uncertain descriptor systems. The sufficient conditions of Chou et al. [1] are the generalized versions of the results given by Lin et al. [7]. Using the singular value decomposition (SVD) and the matrix measure approach, some sufficient conditions are proposed in this paper. These sufficient conditions, respectively, proposed in this paper, by Lin et al. [8,9] and by Chou et al. [1] are derived by different approaches for studying the robust controllability of linear uncertain descriptor systems. So, it is difficult to compare the conservatism by using the mathematical analysis. On the other hand, here it should be noticed that the sufficient conditions of Lin et al. [8,9] can be applied to the case that the matrix $R_0 = [E_n \;\; E_d]$ has no full row rank, while the sufficient conditions proposed in this paper and by Chou et al. [1] only can be used to the case that the matrix $R_0 = [E_n \;\; E_d]$ has full row rank. Therefore, three examples are given in this paper for illustration and to make some numerical comparisons between the proposed sufficient conditions and those of Lin et al. [8,9] and Chou et al. [1] under the assumption that the matrix $R_0 = [E_n \;\; E_d]$ has full row rank. From the above numerical examples, under the assumption that the matrix $R_0 = [E_n \;\; E_d]$ has full row rank, we can see that the proposed sufficient conditions may obtain less conservative results than those of Lin et al. [8,9] and Chou et al. [1]. The reasons why the proposed sufficient conditions are less conservative are: (i) The proposed sufficient conditions take the directional information into consideration. This can be explained by the fact that as a parameter varies in different directions, it affects the system's properties differently. That is, the effect of a single parameter $\alpha$ on the system's properties can be completely different for the same $|\alpha|$ and opposite sign. Therefore, any sufficient conditions, that ignore the signs, may obtain more conservative results. (ii) The singular value decomposition used to derive the proposed sufficient conditions can be exploited to simplify the analysis and to gain insight into the underlying important factors of the matrices $R_0$, $N_0$, $Q_0$ and $M_0$. Therefore, the proposed sufficient conditions may give less conservative results under the assumption that the matrix $R_0 = [E_n \;\; E_d]$ has full row rank. On the other hand, it may be believed that the sufficient conditions proposed in this paper and the sufficient conditions of Lin et al. [8,9] and Chou et al. [1] can be complemented by each other such that the tools of controllability robustness analysis of linear uncertain descriptor systems are more complete.

## 5. Conclusions

In this paper, some sufficient conditions have been established for the robust regularity, the robust I-controllability, the robust R-controllability and the robust C-controllability of the linear descriptor systems with both structured and unstructured parameter uncertainties. The corresponding results for the dual observability robustness problems are straightforward extensions. The results in the proposed theorems have also been extended to obtain another sufficient conditions on the robust regularity, the robust I-controllability, the robust R-controllability and the robust C-controllability for a class of linear descriptor systems with both structure information uncertainties and parameter uncertainties. Three numerical examples have been given to illustrate the applications of the proposed sufficient conditions, and it has been shown that the proposed sufficient conditions could be less conservative than the existing conditions given by Lin et al. [8,9] and Chou et al. [1] under the assumption that the matrix $R_0 = \left[E_n \;\; E_d\right]$ has full row rank. The reasons why the proposed sufficient conditions could be less conservative are also given in Remark 4.

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